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On Almost-Periodic Linear Differential Systems of Millionščikov and Vinograd

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1. INTRODUCTION

In [8], Millionščikov gave an example of a two-dimensional ODE $\dot{x} = B(t)\,x$ with almost-periodic coefficients which is not almost reducible. In [9, 10, 18], Millionščikov and Vinograd discussed two-dimensional, non-almost reducible examples in which $B(t)$ is quasi-periodic with two basic frequencies. In the terminology of Sacker and Sell, these ODEs fail to have discrete spectrum ([12, 15]).

In this paper, we focus attention on the example of Vinograd (and make remarks concerning those of Millionščikov). We show that, for this example, the induced flow on the projective bundle (see Section 2 and [4]) admits an isolated invariant set ([14]). The isolated invariant set has a rather complex structure; it is connected, locally connected at some points, but not locally connected at all points. In this respect, it may be compared with the example of ([3, 14.21–14.24]). We then consider the minimal subset of the projective bundle (it is unique by [5]). We show that this minimal subflow is an almost automorphic extension of the base ([16]). This result illuminates those of [6], where it is shown that almost automorphic extensions are a key to the understanding of the projective flows induced by two-dimensional, almost-periodic linear ODEs.

2. PRELIMINARIES

We introduce notation and review some basic ideas.

2.1. DEFINITIONS. Let C be the space of all continuous mappings from $\mathbb{R}$ to the set of 2×2 real matrices. Give C the topology of uniform convergence on compact sets. The map $\Phi: C \times \mathbb{R} \to C$: $(A, t) \to A_t$, where $A_t(s) = A(t+s)$, defines a real flow ([11]) on C. Suppose $B \in C$ is uniformly bounded and uniformly continuous. Then $\Omega = \mathrm{cls}\{B_t : t \in \mathbb{R}) \subset C$ is compact

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metric, and $\Phi|_{\Omega\times\mathbb{R}}$ defines a flow $(\Omega, \mathbb{R})$. We can "extend B to Ω" as follows: let $b(\omega)=\omega(0)$ $(\omega\in\Omega)$; then $b(\omega_t)=\omega_t(0)=\omega(t)$ $(\omega\in\Omega, t\in\mathbb{R})$. In particular, if $\omega_0\equiv B\in\Omega$, then $b(\omega_0\cdot t)=B(t)$. We call Ω the *hull* of B.

2.2. DEFINITIONS. Let $(X, \mathbb{R})$ be a flow; we denote the "position" of $x\in X$ after "time" $t\in\mathbb{R}$ by $x\cdot t$. Let $Y\subset X$ be a compact invariant set (thus $Y\supset Y\cdot\mathbb{R}=\{y\cdot t: t\in\mathbb{R},\ y\in Y\}$). We say Y is an *isolated invariant set* ([14]) if there is a closed neighborhood N of Y such that Y is the maximal invariant subset of N. We say Y is *minimal* if the orbit $\{y\cdot t: t\in\mathbb{R}\}$ is dense in Y for each $y\in Y$. If Y is metrizable with metric d, then $(Y, \mathbb{R})$ is *almost periodic* if, given $\varepsilon>0$, there is a $\delta>0$ such that $d(y_1, y_2)<\delta\Rightarrow d(y_1\cdot t, y_2\cdot t)<\varepsilon$ for all $t\in\mathbb{R}$. If the function B of Definitions 2.1 is almost periodic in the sense of Bohr, then its hull $(\Omega, \mathbb{R})$ is minimal and almost periodic ([11]). Finally, let $(X, \mathbb{R})$ and $(\Omega, \mathbb{R})$ be flows with X and Ω compact metric, and let $\pi: X\to\Omega$ be a flow homomorphism (thus π is continuous, and $\pi(x\cdot t)=\pi(x)\cdot t$ for all $x\in X$ and $t\in\mathbb{R}$). Say that $(X, \mathbb{R})$ is an *almost automorphic extension* of $(\Omega, \mathbb{R})$ if card $\pi^{-1}(\omega)=1$ for some $\omega\in\Omega$ ([16, 17]).

2.3. DEFINITIONS. Let B, Ω, b be as in Definitions 2.1 (thus B is uniformly bounded and uniformly continuous, and $B(t)$ is 2×2 $(t\in\mathbb{R})$). Consider the ODEs

$$\dot{x}=B(t)\,x, \tag{1}$$

$$x=b(\omega\cdot t)\,x \qquad (\omega\in\Omega). \tag*{$(1)_\omega$}$$

We say that Eq. (1) "induces" Eqs. $(1)_\omega$.

Equations $(1)_\omega$ induce a flow on $\Omega\times\mathbb{R}^2$, as follows: let $(\omega, x_0)\cdot t=(\omega\cdot t, x(t))$, where $x(t)$ satisfies $(1)_\omega$ with initial condition $x(0)=x_0$. The flow $(\Omega\times\mathbb{R}^2, \mathbb{R})$, which we will often denote by L, is a *linear skew-product flow*, or LSPF ([12, 15]). Let $\mathbb{S}^1\subset\mathbb{R}^2$ be the unit circle, and let $\mathbb{P}^1$ be real projective 1-space = the set of all lines through the origin in $\mathbb{R}^2$. Let $\Sigma_s=\Omega\times\mathbb{S}^1$, and let $\Sigma_p=\Omega\times\mathbb{P}^1$. The LSPF L induces flows $(\Sigma_s, \mathbb{R})$ and $(\Sigma_p, \mathbb{R})$ by linearity (see, e.g., [4]). Let $\pi_s: \Sigma_s\to\Omega$ and $\pi_p: \Sigma_p\to\Omega$ be the projections; these maps are *flow homomorphisms* (i.e., $\pi_s(\sigma\cdot t)=\pi_s(\sigma)\cdot t$ if $\sigma\varepsilon\Sigma_s$ and $\pi_p(\sigma\cdot t)=\pi_p(\sigma)\cdot t$ if $\sigma\in\Sigma_p$). We will use the (usual) polar coordinate θ to parameterize $\mathbb{S}^1$, and we will use $\varphi=2\theta$ to parameterize $\mathbb{P}^1$. Thus (ω, θ) will denote a point of Σ_s, and (ω, φ) will denote a point of Σ_p.

2.4. DEFINITIONS. Let L be the LSPF of Definitions 2.3. Say that $\lambda\in\mathbb{R}$ is in the *resolvent* of L ([12]) if the ODE $\dot{x}=[-\lambda I+b(\omega\cdot t)]\,x$ has an exponential dichotomy for some (hence any) $\omega\in\Omega$. Define the *spectrum* of L, Sp(L), to be the complement in $\mathbb{R}$ of the resolvent of L.

Since our differential systems are two-dimensional, the next result is a corollary of the Sacker–Sell spectral theorem ([13, 15]).

2.5. PROPOSITION. *Let L be the LSPF of Definitions* 2.3. *Let* $(\Omega, \mathbb{R})$ *be minimal and almost periodic. Then* $\mathrm{Sp}(L)$ *is* (i) *a single point, or* (ii) *two points, or* (iii) *a nondegenerate closed interval.*

The next proposition is proved in ([15, Theorem 4, p. 185]).

2.6. PROPOSITION. *Let* $x(t)$ *be a non-zero solution to some Eq.* $(1)_\omega$, *and let* $\lambda = \lim_{n\to\infty} (1/t_n)\|x(t_n)\|$ *for some sequence* t_n *such that* $|t_n| \to \infty$. *Then* $\lambda \in \mathrm{Sp}(L)$.

From ([4]), we have the following.

2.7. PROPOSITION. *Let L be the LSPF of Definitions* 2.3, *and suppose* $(\Omega, \mathbb{R})$ *is minimal and almost periodic. Suppose* $\mathrm{Sp}(L)$ *is a nondegenerate closed interval. Then the flow* $(\Sigma_p, \mathbb{R})$ *admits exactly two ergodic measures* ([11]) μ_1 *and* μ_2. *There are disjoint Borel sets* $B_1, B_2 \subset \Sigma_p$ *such that* $\mu_i(B_i) = 1$, *and* $\mathrm{card}(B_i \cap \pi_p^{-1}(\omega)) = 1$ *for* μ_0*–a.a.* $\omega \in \Omega$ $(i = 1, 2)$. *Here* μ_0 *is the unique ergodic measure on* Ω ([11, *Theorem* 9.34, *p.* 510]).

The final proposition is stated in [13] and proved in [5].

2.8. PROPOSITION. *Let L be the LSPF of Definitions* 2.3, *and suppose* $(\Omega, \mathbb{R})$ *is minimal and almost periodic. Suppose* $\mathrm{Sp}(L)$ *is a nondegenerate interval. Then* $(\Sigma_p, \mathbb{R})$ *contains a unique minimal set M. The measures* μ_1 *and* μ_2 *ergodic with respect to* Σ_p *are supported on M.*

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As stated in the Introduction, we will restrict attention to Vinograd's example (actually, class of examples; see [18]). We do this because: (i) his discussion is very clear; (ii) two of our results (Proposition 3.11b and c) do not apply to Millionščikov's limit periodic example ([8]). (They do apply to his quasi-periodic example ([9, 10]).) We first review properties of Vinograd's equation; see ([18]) for details.

3.1. Vinograd's ODE

$$\dot{x} = B(t)\,x = \begin{bmatrix} 0 & 1 + A(t) \\ 1 - A(t) & 0 \end{bmatrix} x$$

is quasi-periodic with two basic frequencies. Hence the hull Ω of B is a two-torus K^2. Let ψ_1, ψ_2 be 1-periodic coordinates on K^2. The flow $(K^2, \mathbb{R})$ is given by $(\psi_1, \psi_2) \cdot t = (\psi_1 + t, \psi_2 + \alpha t)$, where α is irrational. We can assume $0 < \alpha < 1$; see [18].

3.2. The function $B(t)$ is the uniform limit of functions

$$B_n(t) = \begin{bmatrix} 0 & 1 + A_n(t) \\ 1 - A_n(t) & 0 \end{bmatrix}.$$

Let $b_n, b\colon K^2 \to$ set of 2×2 real matrices be the extensions of B_n, B to K^2 (2.1), and write

$$b_n(\omega) = \begin{bmatrix} 0 & 1 + a(\omega) \\ 1 - a(\omega) & 0 \end{bmatrix}, \quad b(\omega) = \begin{bmatrix} 0 & 1 + a(\omega) \\ 1 - a(\omega) & 0 \end{bmatrix} \quad (\omega \in \mathbb{K}^2).$$

In polar coordinates (r, θ) we have

$$\dot{r} = r \sin 2\theta, \qquad \dot{\theta} = -a_n(\omega \cdot t) + \cos 2\theta, \tag*{$(2)_{\omega,n}$}$$

$$\dot{r} = r \sin 2\theta, \qquad \dot{\theta} = -a(\omega \cdot t) + \cos 2\theta. \tag*{$(2)_{\omega}$}$$

The functions a_n, a satisfy

$$0 \leqslant a_n(\omega) \to a(\omega), \qquad \text{and } (a_n(\omega))_{n=1}^{\infty} \text{ is nondecreasing} \qquad (\omega \in K^2). \tag{3}$$

3.3. Let $\omega_0 = (0, 0) \in K^2$. Equations $(2)_{\omega,n}$ have solutions $x_1^n(t) = (r_1^n(t), \theta_1^n(t))$, $x_2^n(t) = (r_2^n(t), \theta_2^n(t))$ such that

$$\lim_{|t| \to \infty} \frac{1}{t} \ln \|x_1^n(t)\| = -\beta_n < \frac{-1}{2}, \quad \lim_{|t| \to \infty} \frac{1}{t} \ln \|x_2^n(t)\| = \beta_n > \frac{1}{2}, \tag{4}$$

$$-\pi/4 < \theta_1^n(t) < \theta_1^{n+1}(t) < \theta_2^{n+1}(t) < \theta_2^n(t) < \pi/4 \qquad (t \in \mathbb{R},\ n \geqslant 1), \tag{5}$$

$$0 < \inf_t (\theta_1^n(t), \theta_2^n(t)) \equiv \gamma_n \to 0 \qquad \text{as} \quad n \to \infty. \tag{6}$$

This completes the review of Vinograd's equation. The following proposition is an easy consequence of the perturbation theorem of ([13]).

3.4. **Proposition.** *Let L be the LSPF on $K^2 \times \mathbb{R}^2$ defined by Eqs. $(2)_\omega$. Then* $\mathrm{Sp}(L)$ *is a nondegenerate closed interval.*

3.5. Notation. Let L_n resp. L be the LSPF induced on $K^2 \times \mathbb{R}^2$ by Eqs. $(2)_{\omega,n}$ resp. Eqs. $(2)_\omega$. For each $n \geqslant 1$, Eqs. $(2)_{\omega,n}$ induce flows on

$\Sigma_s = K^2 \times \mathbb{S}^1$ and $\Sigma_p = K^2 \times \mathbb{P}^1$(Definitions 2.3). If $(\omega, \theta) \in \Sigma_s$ and $t \in \mathbb{R}$, we denote the image of (ω, θ) under the time-t map by $(\omega, \theta)_n t$. Similarly, if $(\omega, \varphi) \in \Sigma_p$, let $(\omega, \varphi)_n t$ be the image of (ω, φ) under the time-t map. In addition, Eqs. $(2)_\omega$ induce flows $(\Sigma_s, \mathbb{R})$ and $(\Sigma_p, \mathbb{R})$; we use $(\omega, \theta) \cdot t$ and $(\omega, \varphi) \cdot t$ in referring to these flows.

3.6. DEFINITIONS. Let ω_0 and $\theta_1^n(t)$, $\theta_2^n(t)$ be as in Definitions 2.3. Define $J_1^n = \text{cls}\{(\omega_0 \cdot t, \theta_1^n(t)) : t \in \mathbb{R}\} \subset \Sigma_s$, $J_2^n = \text{cls}\{(\omega_0 \cdot t, \theta_2^n(t)) \subset \Sigma_s$. Then J_1^n are compact, disjoint (by (6)), invariant, and are subsets of $K^2 \times [-\pi/4, \pi/4]$ (by (5)). Hence, by Propositions 2.8 and 2.6, $\text{Sp}(L_n)$ must be a single point, or two points. It follows from Proposition 2.6 and (4) that $\text{Sp}(L_n) = \{-\beta_n, \beta_n\}$. By (5) and the Sacker–Sell spectral theorem (see also the discussion beginning Section 3 in [4]), $\text{card}(J_1^n \cap \pi_s^{-1}(\omega)) = 1$ for all $\omega \in \Omega$ $(i = 1, 2; n \geqslant 1)$. Hence we can define continuous functions g_n and h_n: $K^2 \to [-\pi/4, \pi/4] \subset \mathbb{S}^1$ as follows: $J_1^n \cap \pi_s^{-1}(\omega) = (\omega, g_n(\omega))$; $J_2^n \cap \pi_s^{-1}(\omega) = (\omega, h_n(\omega))$ $(n \geqslant 1)$. We have

$$(\omega \cdot t, g_n(\omega \cdot t)) = (\omega, g_n(\omega))_n t, \tag{7}$$

$$(\omega \cdot t, h_n(\omega \cdot t)) = (\omega, h_n(\omega))_n t \qquad (\omega \in K^2, t \in \mathbb{R}). \tag{8}$$

By (5) and (6), we also have

$$-\pi/4 < g_n(\omega) \leqslant g_{n+1}(\omega) < h_{n+1}(\omega) \leqslant h_n(\omega) < \pi/4 \qquad (n \geqslant 1, \omega \in K^2). \tag{9}$$

3.7. LEMMA. (a) *Let* $g(\omega) = \lim_{n \to \infty} g_n(\omega)$, $h(\omega) = \lim_{n \to \infty} h_n(\omega)$. *Then* $(\omega \cdot t, g(\omega \cdot t)) = (\omega, g(\omega)) \cdot t$ *and* $(\omega \cdot t, h(\omega \cdot t)) = (\omega, h(\omega)) \cdot t$. (*For the meaning of the dot* $\cdot$, *see Notation* 3.5).

(b) *The set* $J = \{(\omega, \theta) \in \Sigma_s : g(\omega) \leqslant \theta \leqslant h(\omega)\}$ *is compact, invariant, and is the maximal invariant subset of* $K^2 \times [-\pi/4, \pi/4]$ *with respect to the flow induced by Eqs.* $(2)_\omega$. *Thus* J *is an isolated invariant set for this flow* (2.2).

Proof. (a) This is a consequence of (3).

(b) Invariance follows from (a). If (ω_k, θ_k) is a sequence in J, and if $(\omega_k, \theta_k) \to (\omega, \theta)$, then $g_n(\omega_k) \leqslant \theta_k \leqslant h_n(\omega_k)$ for all n and k. Hence $g_n(\omega) \leqslant \theta \leqslant h_n(\omega)$ for all n, so $(\omega, \theta) \in J$. Thus J is compact. To see that J is isolated, let $(\omega, \theta) \in T^2 \times [-\pi/4, \pi/4]$. If $\theta < g(\omega)$, then $\theta < g_n(\omega)$ for some n. Choose θ_n such that $\theta < \theta_n < g_n(\omega)$. Let $\theta_n(t)$ resp. $\theta(t)$ be the solution to $(2)_{\omega,n}$ resp. $(2)_\omega$ such that $\theta_n(0) = \theta_n$ resp. $\theta(0) = \theta$. By (3), $\theta(t) \leqslant \theta_n(t)$ for all $t \geqslant 0$. By (4) and the Sacker–Sell spectral theorem, $\theta_n(t) < -\pi/4$ for some $t > 0$. Hence $\theta(t) < -\pi/4$ for some $t > 0$. In a similar way, if $\theta > h(\omega)$, then $\theta(t) > \pi/4$ for some $t < 0$. Hence J is isolated.

3.8. ***Remarks.*** (a) The set $J' = \{(\omega, \theta + \pi): (\omega, \theta) \in J\}$ is also an isolated invariant set.

(b) Let J_p be the projection of J to Σ_p. Then L_p is the unique isolated invariant subset of Σ_p. Also, J_p is homeomorphic to J, and in fact $(J_p, \mathbb{R})$ and $(J, \mathbb{R})$ are isomorphic as transformation groups.

We now discuss the structure of J.

3.9. PROPOSITION. *There is a residual subset $\Omega_1 \subset K^2$ such that* $\omega \in \Omega_1 \Rightarrow g(\omega) = h(\omega)$ (i.e., $\operatorname{card}(J \cap \pi_s^{-1}(\omega)) = 1$).

Proof. First note that, by (6), there is at least one such ω. Let 2^{Σ_s} be the set of nonempty compact subsets of Σ_s with the Hausdorff metric ρ ([1, p. 112]; ρ is defined using some metric on Σ_s). Define $\tau: K^2 \to 2^{\Sigma_s}$: $\tau(\omega) = J \cap \pi^{-1}(\omega)$. Then τ is lower semicontinuous, hence has a residual set of continuity points ([1, p. 112, 114]). Let $\tilde{\omega}$ be a continuity point of τ. If $\tau(\tilde{\omega})$ contains more than one point, then there is a neighborhood $N(\tilde{\omega}) \subset K^2$ and a $\delta > 0$ such that $\omega \in N(\tilde{\omega}) \Rightarrow \operatorname{diam} \tau(\omega) \geqslant \delta > 0$. Since $(K^2, \mathbb{R})$ is minimal, there exist times $t_1, ..., t_k$ such that $\bigcup_{i=1}^{k} N(\tilde{\omega}) \cdot t_i = K^2$. By invariance of J, $\exists\, \delta' > 0$ such that $\operatorname{diam} \tau(\omega) \geqslant \delta' > 0$ for *all* $\omega \in \Omega$. This contradicts the first sentence of the proof, so card $(J \cap \pi^{-1}(\omega)) = 1$ for each continuity point $\tilde{\omega}$ of τ.

3.10. PROPOSITION. *There is a set $\Omega_2 \subset K^2$ such that:*

(i) $\mu_0(\Omega_2) = 1$; (ii) $\omega \in \Omega_2 \Rightarrow J \cap \pi_s^{-1}(\omega)$ *is a nondegenerate interval. As before, μ_0 is the unique ergodic measure on Ω.*

Proof. There are two measures, υ_1 and υ_2, supported on J and ergodic with respect to $(J, \mathbb{R})$ (Propositions 2.7, 2.8, and Remark 3.8b). Also, there are invariant Borel sets $B_1, B_2 \subset J$ such that $B_1 \cap B_2 = \phi$, $\upsilon_1(B_1) = \upsilon_2(B_2) = 1$, and $\operatorname{card}(B_i \cap \pi_s^{-1}(\omega)) = 1$ for μ_0–a.a. $\omega \in K^2$ (Proposition 2.7). Let $\Omega_2 = \pi_s(B_1) \cap \pi_s(B_2)$. Then $\mu_0(\Omega_2) = 1$, and $\omega \in \Omega_2 \Rightarrow J \cap \pi_s^{-1}(\omega)$ contains two points. Now Proposition 3.10 follows from the definition of J.

The proposition to follow gives an indication of the remarkably complicated nature of the flow $(\Sigma_p, \mathbb{R})$.

3.11. PROPOSITION. (a) *J is connected,*

(b) *J is locally connected at all points $(\omega, \theta) \in J$ such that* $\operatorname{card}(J \cap \pi^{-1}(\omega)) = 1$,

(c) *J is not locally connected at all points.*

Proof. (a) Combine the following facts: (i) K^2 is connected; (ii) $J \cap \pi^{-1}(\omega)$ is connected for all $\omega \in K^2$, (iii) Ref. [2, Chap. VI, 3.4].

(b) Let $\{q\} = J \cap \pi^{-1}(\omega)$. By ([7, p. 99]), any neighborhood W of q contains a neighborhood V of q such that: (*i*) $\pi(V)$ is open in K^2; (ii) $\pi^{-1}\pi(V) = V$. We can assume $\pi(V)$ is connected. By ([2, Chap. VI, 3.4]), V is connected.

(c) Suppose J is everywhere locally connected. By ([19, Theorem 4.1, p. 27]), each pair of points in J may be joined by a simple arc (a homeomorph of $[0, 1]$). Let $Z = \{(\omega, \theta) \in J: \{(\omega, \theta)\} = \pi_s^{-1}(\omega) \cap J\}$. Pick $z_0 \in Z$, let y be any point in J not on the orbit through z_0, and let $\tilde{\eta}: [0, 1] \to J$ be a simple arc with $\tilde{\eta}(0) = z$, $\tilde{\eta}(1) = y$.

As in 3.1, let ψ_1, ψ_2 be one-periodic coordinates on K^2. To simplify things, we suppose $\pi_s(z_0) = (0, 0) \in K^2$; this will not affect the generality of the proof. Let $l = \{(\psi_1, 0): -1/4 < \psi_1 < 1/4\}$, and let $Q = l \cdot (-1/4, 1/4) = \{(\psi_1 + t, \alpha t): -1/4 < t < 1/4, -1/4 < \psi_1 < 1/4\}$. (Recall $0 < \alpha < 1$; see 3.1.) Let $r_0 = \sup\{r \in [0, 1]: \eta(r)$ is in the orbit through z_0 for all $0 \leqslant r \leqslant r_0\}$. Then $r_0 < 1$. There is a unique $T \in \mathbb{R}$ such that $\tilde{\eta}(r_0) = z_0 \cdot T$. Define $\eta(r) = [\tilde{\eta}(r + r_0) \cdot (-T)]$ $(0 \leqslant r \leqslant 1 - r_0)$. Then choose $\varepsilon_0 > 0$ such that (i) $\pi_s \circ \eta([0, \varepsilon_0]) \subset Q$; (ii) $\eta(\varepsilon_0)$ is not on the orbit through z_0. Finally, define the "projection" σ of η onto $\pi_s^{-1}(l) \subset \pi_s^{-1}(Q)$ as follows: $\sigma(r) = \eta(r) \cdot t_r$, where t_r is the unique element of $[-1/4, 1/4]$ such that $\eta(r) \cdot t_r \in \pi_s^{-1}(l)$. Thus σ maps $[0, \varepsilon_0]$ to $\pi_s^{-1}(l)$, $\sigma(0) = z_0$, σ is continuous, and (from (ii)) $\sigma(0) \neq \sigma(\varepsilon_0)$. Since $z_0 \in Z$, $\pi_s \sigma(0) \neq \pi_s \sigma(\varepsilon_0)$.

We now forget about $\tilde{\eta}$ and η, and focus attention on σ. We will show that the properties of σ just stated contradict certain properties of J.

First, let M be the unique minimal subset of J (Proposition 2.8 and Remark 3.8b). Then cls $Z = M$. It is also the case that, if $M_l = \text{cls}(Z \cap \pi_s^{-1}(l))$, then

$$M_l \supset M \cap \pi_s^{-1}(l). \tag{10}$$

For, let $m \in M \cap \pi_s^{-1}(l)$. There is a sequence z_n in Z such that $z_n \to m$. There is a sequence $t_n \to 0$ such that, for large n, $z_n \cdot t_n \in \pi_s^{-1}(l)$. Since Z is invariant, $z_n \cdot t_n \in Z \cap \pi_s^{-1}(l)$. Since $z_n \cdot t_n \to m$, m is in M_l. Hence (10) is true.

Next, observe that, in the proof of 3.10, the two Borel sets B_1, B_2 may be assumed to be subsets of M (use the last sentence in Proposition 2.8). Hence $\{\omega \in K^2: \text{card}(M \cap \pi_s^{-1}(\omega)) > 1\}$ has μ_0-measure 1 (and is therefore nonempty). It is clearly invariant. Hence

$$\text{card}(M \cap \pi_s^{-1}(l)) > 1 \quad \text{for uncountably many } \omega \text{ in any nondegenerate subinterval of } l. \tag{11}$$

Here we abuse language slightly, and speak of l as an interval (parametrized by ψ_1).

We can now piece things together. Let $U = \{\omega \in l: \sigma(r) \in \pi_s^{-1}(\omega)$ for all r in some nondegenerate subinterval $[a_\omega, b_\omega]$ of $[0, \varepsilon_0]\}$. Then U is countable. Let $I_0 = \pi_s \circ \sigma([0, \varepsilon_0]) \subset l$; then I is connected, hence is a subinterval of the "interval" l. It is a *nondegenerate* subinterval, because $\pi_s \circ \sigma(\varepsilon_0) \neq \pi_s \circ \sigma(0)$. Order I_0 so that $\pi_s \circ \sigma(0) < \pi_s \circ \sigma(\varepsilon_0)$, and let $I = [\pi_s \circ \sigma(0), \pi_s \circ \sigma(\varepsilon_0)] \subset l$. Using (11), we may choose $\omega_0 \in \text{interior } (I)$ such that $\omega_0 \notin U$ and $\operatorname{card}(M \cap \pi_s^{-1}(\omega_0)) > 1$. Let $r_0 = \sup\{r \in [0, \varepsilon_0]:$ for all $0 \leqslant r' \leqslant r$, $\pi_s \circ \sigma(r') \leqslant \omega_0\}$. Then $\pi_s \circ \sigma(r_0) = \omega_0$. Define $z_0 = \sigma(r_0)$.

To complete the proof, pick any sequence z_n in $Z \cap \pi_s^{-1}(l)$ such that $\pi_s(z_n) \to \omega_0$. For each sufficiently large n, we can choose r_n such that (i) $\sigma(r_n) = z_n$; (ii) $r_n \to r_0$. Since σ is continuous, $\sigma(r_n) \to \sigma(r_0) \Rightarrow z_n \to z_0$. By (10), we have $\{z_0\} = M \cap \pi_s^{-1}(\omega_0)$. This contradicts our choice of ω_0. Hence J cannot be everywhere locally connected. The proof of Proposition 3.11 is completed.

Having discussed J in some detail, we now turn to its minimal subset M, and to the unique minimal subset M_p of Σ_p. By Remark 3.8b, the flows $(M, \mathbb{R})$ and $(M_p, \mathbb{R})$ are flow isomorphic.

3.12. PROPOSITION. *If M_p is the unique minimal subset of Σ_p, then $(M_p, \mathbb{R})$ is an almost automorphic extension of $(K^2, \mathbb{R})$. However, $(M_p, \mathbb{R})$ is not an almost-periodic flow.*

Proof. The first sentence follows from Remark 3.8b and Proposition 3.9. The second sentence follows from the fact that M_p supports more than one invariant measure, and ([11, Theorem 9.34, p. 510]).

3.13. *Remarks.* (a) We can obtain more information about J and its minimal subset M by applying the theory of Selgrade ([14]). By his results, one endpoint of a nondegenerate interval $J \cap \pi_s^{-1}(\omega)$ is a positive expansion point, and the other endpoint is a negative expansion point (for definitions see [14]). Using this fact and results of ([4, Section 3]), one can show that one ergodic measure υ_1 on J assigns measure 1 to the set of "upper endpoints" $\{(\omega, h(\omega)): \omega \in K^2\}$, while the other ergodic measure υ_2 assigns measure 1 to the set of "lower endpoints" $\{(\omega, g(\omega)): \omega \in K^2\}$. Since υ_1 and υ_2 are supported on M, we see that M contains "μ_0-almost all" of the endpoints of the nondegenerate intervals in J.

(b) Both of Millionščikov's examples generate flows $(\Sigma_p, \mathbb{R})$ which contain isolated invariant sets, and the methods used here may be applied to both of them. All the results of Section 3 are true for his quasi-periodic example ([9]). All the results of Section 3 except Proposition 3.11b and c are true for his limit-periodic example ([8]). These latter propositions fail because the hull is not locally connected.

(c) It would be of interest to know whether $M = J$ for the quasi-periodic examples ([9, 18]). This is true for the limit-periodic example ([8]).

(d) It would also be of interest to know whether every two-dimensional, almost-periodic ODE with interval spectrum gives rise to a flow $(\Sigma_p, \mathbb{R})$ containing an isolated invariant set with the structure outlined here.

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